$$\text{mod}\,(1+\frac{z}{m+1})^m < \text{mod}\,(1+\frac{z}{n})^n < \text{mod}\,(1+\frac{z}{m})^{m+1}, \tag{1}$$

$$\arg\,(1+\frac{z}{m+1})^m < \arg\,(1+\frac{z}{n})^n < \arg\,(1+\frac{z}{m})^{m+1}. \tag{2}$$

Now $\underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m})^{m+1}$, since

$$\underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m+1})^m = \frac{\underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m+1})^{m+1}}{\underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m+1})} = f(z), \text{ and}$$

$$\underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m})^{m+1} = \underset{m \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{m})^m\,(1+\frac{z}{m}) = f(z).$$

Hence, since when two complex numbers are equal their moduli and arguments are equal,

$$\underset{m \doteq \infty}{\mathrm{L}}\,\text{mod}\,(1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}}\,\text{mod}\,(1+\frac{z}{m})^{m+1} = \text{mod}\,f(z),$$

$$\underset{m \doteq \infty}{\mathrm{L}}\,\arg\,(1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}}\,\arg\,(1+\frac{z}{m})^{m+1} = \arg f(z).$$

Therefore, from the preceding inequalities, since $\text{mod}\,(1+\frac{z}{n})^n$ lies between two numbers, both of which have the same limit $\text{mod}\,f(z)$

$$\underset{n \doteq \infty}{\mathrm{L}}\,\text{mod}\,(1+\frac{z}{n})^n = \text{mod}\,f(z).$$

Similarly, $\underset{n \doteq \infty}{\mathrm{L}}\,\arg\,(1+\frac{z}{n})^n = \arg f(z)$.

Therefore, $\underset{n \doteq \infty}{\mathrm{L}}\,(1+\frac{z}{n})^n = f(z)$.

§4. CASE 2. $180° > \text{ARG } z > 90°$.

With the same notation as before:

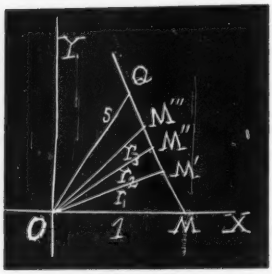

Fig. 2.

$$\phi_1 < \phi_2 < \phi_3, \quad m\phi_1 < n\phi_2 < (m+1)\phi_3,$$

and as before:

$$\underset{n \doteq \infty}{\mathrm{L}} \arg(1+\frac{z}{n})^n = \arg f(z).$$

But we can not be sure that the inequalities (1) hold. We can however proceed as follows: By Trigonometry, Fig. 2, where length $MQ = \frac{r}{\lambda}$,

$$[\mathrm{mod}(1+\frac{z}{\lambda})]^2 = s^2 = 1^2 + (\frac{r}{\lambda})^2 + \frac{2r\cos\phi}{\lambda} = 1 + \frac{r}{p}, \text{ where } p = \frac{\lambda}{2\cos\phi + r/\lambda}.$$

$[\mathrm{mod}(1+\frac{z}{\lambda})^\sigma]^2 = [(1+\frac{r}{p})^p]^{\sigma/p}$. If now σ and p are two real numbers which become indefinitely great in any way such that the difference between σ and λ remains finite then we have:

$$\underset{\lambda \doteq \infty}{\mathrm{L}} (\frac{\sigma}{\lambda}) = 1. \quad p \doteq \infty, \text{ as } \lambda \doteq \infty. \quad \underset{\lambda \doteq \infty}{\mathrm{L}} \frac{\sigma}{p} = 2\cos\phi.$$

$$\underset{\lambda \doteq \infty}{\mathrm{L}} [\mathrm{mod}(1+\frac{z}{\lambda})^\sigma]^2 = e^{2r\cos\phi}.$$

Taking the square root we have:

$$\underset{\lambda \doteq \infty}{\mathrm{L}} \mathrm{mod}(1+\frac{z}{\lambda})^\sigma = e^{r\cos\phi} = e^{\mathrm{mod}(z)\cos\arg(z)},$$

a result that is independent of λ and σ.

As special cases, if (λ, σ) have the pairs of values $(m+1, m)$, (n, n), $(m, m+1)$, (m, m), we see that the three numbers of the inequalities (1) have the same limit, and that this limit is mod $f(z)$.

Since the modulus and argument of $(1+\frac{z}{n})^n$ have for limit, respectively, the modulus and argument of $f(z)$, we have:

$$\underset{n \doteq \infty}{\mathrm{L}} (1+\frac{z}{n})^n = f(z).$$

This method might also have been used in case 1.

§5. CASE 3. WHEN 360°>ARG z>180°.

We may fold over the figure on the axis of x, and consider the conjugate. Thus by cases 1 or 2:

$$\underset{n\doteq\infty}{\mathrm{L}}\,(1+\frac{z'}{n})^n=f(z'),$$

where z' is the conjugate of z; *i. e.*,

$$\underset{n\doteq\infty}{\mathrm{L}}\,(1+\frac{x-yi}{n})^n=f(x-yi).$$

As the real and the imaginary parts are respectively equal on the two sides of this equality, the equality will remain if i is changed into $-i$. Hence

$$\underset{n\doteq\infty}{\mathrm{L}}\,(1+\frac{x+yi}{n})^n=f(x+yi),\text{ or }\underset{n\doteq\infty}{\mathrm{L}}\,(1+\frac{z}{n})^n=f(z).$$

§6. THE EXTENSION TO A NEGATIVE n.

Let $n=-p$, then

$$\underset{n\doteq\infty}{\mathrm{L}}\,(1+\frac{z}{n})^n=\underset{p\doteq\infty}{\mathrm{L}}\,(1-\frac{z}{p})^{-p}=\underset{p\doteq\infty}{\mathrm{L}}\,(1+\frac{z}{p-z})^p$$

$$=\underset{p\doteq\infty}{\mathrm{L}}\,(1+\frac{z}{\rho(\cos\alpha+i\sin\alpha)})^p=\underset{p\doteq\infty}{\mathrm{L}}\,[(1+\frac{z(\cos\alpha-i\sin\alpha)}{\rho})^\rho]^{p/\rho}$$

$$=f\,[z\mathrm{L}(\cos\alpha-i\sin\alpha)]=f(z),\text{ since}$$

$$p-z=p-x-yi=\rho(\cos\alpha+i\sin\alpha),\quad \rho=[(p-x)^2+y^2]^{\frac{1}{2}},$$

$$\underset{p\doteq\infty}{\mathrm{L}}\,\frac{p}{\rho}=1,\quad \underset{p\doteq\infty}{\mathrm{L}}\,\alpha=0,\text{ and }\underset{p\doteq\infty}{\mathrm{L}}\,(\cos\alpha-i\sin\alpha)=1.$$

Hence in all cases when z is complex and n is real

$$F(z)=f(z).$$

§7. THE CASE WHEN z AND n ARE BOTH COMPLEX.

We quote from the text p. 240: "If both z and n are complex, we have, if $n=m(\cos\phi+i\sin\phi)$,

$$\lim_{n\doteq\infty}(1+\frac{z}{n})^n=\lim_{m\doteq\infty}(1+\frac{z}{m(\cos\phi+i\sin\phi)})^{m(\cos\phi+i\sin\phi)}$$

$$=\lim_{m\doteq\infty}[(1+\frac{z(\cos\phi-i\sin\phi)}{m})^m]^{\cos\phi+i\sin\phi}=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}.$$

This is as far as we can carry the proof. If, however, we agree to give to a complex exponent such an interpretation that the third property, viz: $f(z)=[f(1)]^z$, shall still hold even when z is complex, we have

$$F(z)=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}=f(1)^z=f(z).$$

Thus for all values of x and n, we have

$$F(x)=f(x).$$

That the series denoted by $f(x)$ is convergent has been seen from the mode of its derivation, since each of the constituent series of which it is composed is convergent whether x be real or complex. The result

$$\lim_{n\doteq\infty}(1+\frac{x}{n})^n=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

is known as the Exponential Theorem."

SYRACUSE UNIVERSITY, *12 April, 1909.*

RATIONAL EDGED CUBOIDS WITH EQUAL VOLUMES AND EQUAL SURFACES.

By L. E. DICKSON, The University of Chicago.

1. In the BULLETIN, May, 1909, p. 401, Professor Kasner proposed the problem to find two cuboids (rectangular parallelopipeds) with equal volumes and equal surfaces, and in which the dimensions are all integral.

The problem is to find two distinct triples of integers such that

$$xyz=x'y'z', \quad xy+xz+yz=x'y'+x'z'+y'z'. \tag{1}$$

I shall prove that each integer must exceed unity and that the volume xyz must be the product of five or more primes. Among the solutions which I obtain, the simplest are

$$p,\ p,\ \tfrac{1}{2}(p+1)p^2;\quad p^2,\ p^2,\ \tfrac{1}{2}(p+1) \qquad (p \text{ odd}), \tag{2}$$

$$p,\ (p+1)q,\ (p+1)(p+1-q);\quad p+1,\ (p+1)p,\ q(p+1-q) \quad (1<q<p), \tag{3}$$

in which $q=r$ and $q=p+1-r$ give the same sets. While for (3) the volume contains at least six primes, for (2) there may be only five primes, for example,

$$3,\ 3,\ 18;\ 9,\ 9,\ 2. \qquad 5,\ 5,\ 75,\ 25,\ 25,\ 3. \tag{2'}$$

The simplest examples under (3) are

$$3,\ 8,\ 8;\ 4,\ 12,\ 4. \qquad 4,\ 10,\ 15;\ 5,\ 20,\ 6. \tag{3'}$$

Generalizations of (2) and (3) are given by (9)-(10) and (16)-(17).

The only solutions in which $xyz<200$ are the first solutions under (2') and (3').

In §§7-9, I show that for three rational edged cuboids with equal volumes and equal surfaces, the volume is the product of six or more primes, that no two of the triples are of the type $p_1,\ p_2,\ p_3p_4p_5p_6$, and that the cases in which two of the triples are $p_1,\ p_2,\ p_3p_4p_5p_6$ and $p_3,\ p_1p_2,\ p_4p_5p_6$ are excluded.

2. Theorem. *Two triples are identical if they have an element in common.*

If $x=x'$, equations (1) reduce to $yz=y'z',\ y+z=y'+z'$.

3. Theorem. *In any solution each integer exceeds unity.*

Two sets with equal products and having an element unity must be of the form

(4) $$1,\ f_1f_2f_3,\ f_4f_5f_6;\ f_1f_4,\ f_2f_5,\ f_3f_6.$$

Denote the corresponding surfaces by $2S_1$ and $2S_2$. Then

$$S_1=f_1f_2f_3+f_4f_5f_6+f_1\ldots f_6, \quad S_2=f_1f_2f_4f_5+f_1f_3f_4f_6+f_2f_3f_5f_6.$$

First, let each $f_i>1$. Since the substitution (12)(45) leaves the triples (4) unaltered, we may set $f_4\geq f_5$.

Now $(f_i-1)(f_j-1)\geqq 1$, so that $f_if_j\geq f_i+f_j$. Thus

$$f_1\ldots f_6\geqq f_1f_2f_4f_6(f_3+f_5)\geqq f_3f_4f_6(f_1+f_2)+f_1f_2f_5(f_4+f_6).$$

Also $f_3f_4f_6f_2\geq f_2f_3f_5f_6$. Hence $f_1\ldots f_6>S_2$, $S_1>S_2$.

Second, let a single f_i be unity. We may set $f_3=1$, $f_5\geq f_4$. Then

$$f_1\ldots f_6\geq f_1f_2f_5(f_4+f_6)\geq f_1f_2f_4f_5+f_5f_6(f_1+f_2)\geq S_2,$$

since $f_1f_5f_6\geq f_1f_4f_6$. Hence $S_1>S_2$.

Third, let at least two f_i equal unity. If two such f's belong to the same product in (4_1), we may set $f_2=f_3=1$. Then

$$S_1=f_1+f_4f_5f_6+f_1f_4f_5f_6>f_5f_6+f_1f_4(f_5+f_6), \quad S_1>S_2,$$

if $f_5>1$, $f_6>1$. If $f_5=1$ or $f_6=1$, the triples have the common element 1 and are identical by §2. Next, let the two f's which equal unity belong to different products in (4_1). A case like $f_3=f_6=1$ is excluded by §2. Hence we may set $f_3=1$, $f_4=1$. Thus

$$S_1-S_2=f_2f_5(f_1f_6-f_1-f_6)+f_1f_2+f_5f_6-f_1f_6.$$

If $f_2>1$ or $f_5>1$, $S_1-S_2>(f_2f_5-1)(f_1f_6-f_1-f_6)$, so that $S_1-S_2>0$ unless $f_1=1$ or $f_6=1$. In the latter case the triples (4) have a common element 1. The same is true in the remaining case $f_2=f_5=1$.

4. Theorem. *The volume must contain at least five prime factors.*

By §3, each element exceeds 1. Hence there occur four or more primes. If there are just four primes, the triples without a common element (§2) may be designated

(5) $$p_1,\ p_2,\ p_3p_4;\ \ p_3,\ p_4,\ p_1p_2,$$

p_4 being the greatest p, and $p_2\geq p_1$. Then $S_1+p_1p_2p_3p_4=S_2+p_1p_2p_3p_4$ may be written

$$p_1p_2(p_3-1)(p_4-1)=p_3p_4(p_1-1)(p_2-1).$$

Hence $p_4=p_2$, so that the triples (5) are identical by §2.

5. For five primes, each triple must be of the form (1, 1, 3) or (1, 2, 2), the notation indicating the number of prime factors of each element. Consider first

$$(6) \qquad p_1,\ p_2,\ p_3p_4p_5;\ \ p_3,\ p_1p_2,\ p_4p_5,$$

where $p_1 \geq p_2$. From $S_1+p_1\ldots p_5=S_2+p_1\ldots p_5$, we get

$$p_1p_2(p_3-1)(p_4p_5-1)=p_3p_4p_5(p_1-1)(p_2-1).$$

Hence p_1 divides $p_3p_4p_5$. But $p_1\neq p_3$ by §2. Since (6) is unaltered by the interchange of p_4 and p_5, we may set $p_1=p_4$. Thus

$$(7) \qquad p_2(p_3-1)(p_1p_5-1)=p_3p_5(p_1-1)(p_2-1) \quad (p_1\geq p_2).$$

If p_2 does not divide p_1-1, then $p_2=p_5$, and (7) becomes

$$(8) \qquad p_3(p_1+p_2-2)=p_1p_2-1,\ \ (p_3-1)^2=(p_1-p_3)(p_2-p_3).$$

Let g be the greatest common divisor of $p_1-p_3=ga^2$, $p_2-p_3=gb^2$. Then $p_3-1=gab$,

$$(9) \quad p_1=1+gab+ga^2,\ \ p_2=1+gab+gb^2,\ \ p_3=1+gab \ (a,\ b \text{ relatively prime})$$

For these values we have the solution

$$(10) \qquad p_1,\ p_2,\ p_1p_2p_3;\ p_3,\ p_1p_2,\ p_1p_2.$$

For $a=b$, then $a=1$, $p_3=\frac{1}{2}(p_1+1)$ and (10) becomes solution (2). For the remaining cases we may set $a>b$. Examples when the p's are all primes are (2′) and

$$a=2,\ b=1,\ g=2 \text{ or } 6,\ p_1,\ p_2,\ p_3=13,\ 7,\ 5 \text{ or } 37,\ 19,\ 13;$$
$$a=3,\ b=2,\ g=10,\ p_1=151,\ p_2=101,\ p_3=61.$$

Next, let p_2 divide p_1-1. Then (7) is equivalent to

$$(11) \qquad p_1-1=cp_2,\ \ p_3-1=kp_5,\ \ p_1p_5-1=lp_3,\ \ lk=c(p_2-1).$$

From the second and third we eliminate p_3 and see that $l=-1+mp_5$, $m=p_1-lk$. Hence by the fourth and first, $m=1+c$. Hence (11_3) may be replaced by

(12) $$l=-1+(1+c)p_5.$$

Let g be the greatest common divisor of c and k. Then, by (11_4),

(13) $c=g\gamma,\ k=g\mu,\ l=\lambda\gamma,\ p_2-1=\lambda\mu$ (λ and μ relatively prime).

Set $\lambda-gp_5=\rho$. Then (12) becomes $\lambda\rho=p_5-1$. We may thus eliminate p_5 and λ:

(14) $p_5=1+\gamma\rho,\ p_2=1+\lambda\mu,\ p_1=1+g\gamma p_2,\ p_3=1+g\mu p_5\quad(\lambda=g+\rho+g\rho\gamma).$

For any positive integers g, ρ, γ, μ, of which the last two are relatively prime, formulae (14) *give values of the p_i leading to a solution**

(15) $$p_1,\ p_2,\ p_1p_3p_5;\ p_3,\ p_1p_2,\ p_1p_5.$$

For $\gamma=\mu=1$, $p_2=(1+g)(1+\rho)$ is composite. For $\gamma=1$, $\mu=2$, $\rho=1$, the least value of g giving prime p's is $g=10$ and $p_5=2$, $p_2=43$, $p_1=431$, $p_3=41$.

6. For two triples of type (1, 2, 2), it suffices to consider

(16) $p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5\ (p_2>p_1;\ p_5\neq p_1,\ p_2;\ p_3\neq p_4).$

Then $S_1=S_2$ gives $p_2p_3p_5(p_4-1)\equiv 0 \pmod{p_1}$. If $p_1=p_3$, $S_1=S_2$ gives

$$p_1p_2+p_4p_5+p_2p_4p_5=p_2p_4+p_2p_5+p_1p_4p_5.$$

Thus $p_4p_5(p_1-1)\equiv 0 \pmod{p_2}$, so that $p_2=p_4$. The middle terms in (16) would then be equal. This case is thus excluded by §2. Hence $p_4\equiv 1 \pmod{p_1}$.

Next, $S_1=S_2$ gives $p_1p_4p_5(p_3-1)\equiv 0 \pmod{p_2}$. First, let p_3-1 be prime to p_2, so that $p_4=p_2$. Removing the factor p_2 from $S_1=S_2$, we get

$$p_1(p_2-p_3-p_5)=p_3p_5(p_2-p_1-1).$$

Since $p_2\equiv 1 \pmod{p_1}$, $p_2-p_1-1=cp_1$, where c is an integer ≥ 0. Thus

(17) $$p_4=p_2=(c+1)p_1+1=p_3+p_5+cp_3p_5.$$

For $c=0$, we set $p_1=p$, $p_3=q$ and obtain solution (3); note that p occurs in the second triple if and only if $q=1$ or p.

*To show that these triples are distinct, it suffices to prove that p_1 does not occur in the second. But $p_1<p_1p_2$, $p_1<p_1p_5$. If $p_1=p_2$, then $\gamma p_2=\mu p_5$. But γ and μ are relatively prime, and γ does not divide p_5 by (14_1).

For $c=1$, $2(p_1+1)=(p_3+1)(p_5+1)$. If $p_3=2$, then $p_1=3k-1$, $p_5=2k-1$. The lowest values of k leading to prime values for each p_i are $k=2, 4, 10$:

$$p_1,\ p_5,\ p_2=5,\ 3,\ 11;\ 11,\ 7,\ 23;\ 29,\ 19,\ 59.$$

The first gives the solution 5, 22, 33; 6, 11, 55. If p_3 is odd, then

$$p_3=2k-1,\ p_1=k(p_5+1)-1,\ p_4=p_2=2k(p_5+1)-1. \tag{18}$$

The p_i are all primes for $k=2$, $p_5=5, 11$; $k=3$, $p_5=7, 13$:

$$p_5,\ p_3,\ p_1,\ p_2=5,\ 3,\ 11,\ 23;\ 11,\ 3,\ 23,\ 47;\ 7,\ 5,\ 23,\ 47;\ 13,\ 3,\ 41,\ 83.$$

For $c=2$, we have $p_3=3$, $p_5=3k+1$, $p_1=7k+3$, or

$$p_3=3l+1,\ p_1=l+p_5+2lp_5,\ \text{or}\ p_3=3l+2,\ p_5=3m+1,\ p_1=6lm+3l+5m+2.$$

But for c even, the p's in (17) are not all primes if $p_1\neq 2$.

In view of the variety of solutions obtained, further cases are not considered.

7. Do there exist three distinct cuboids with rational edges having equal volumes and equal surfaces? We show that there is no solution in which the volume is the product of fewer than six primes. In view of §4, we consider the case of three triples of five primes p_i. Not all three are of the type (1, 1, 3). Let two be of type (1, 1, 3) and one of type (1, 2, 2); it suffices to consider

$$p_1,\ p_2,\ p_3p_4p_5;\ p_3,\ p_4,\ p_1p_2p_5;\ p_5,\ p_1p_2,\ p_3p_4,\ \text{or}\ p_5,\ p_1p_3,\ p_2p_4. \tag{19}$$

In the first alternative, we reduce the S_i modulo p_1 and get

$$p_2p_3p_4p_5\equiv p_3p_4\equiv p_3p_4p_5\pmod{p_1}.$$

Since p_3p_5 is not divisible by p_1, $p_5\equiv 1$, $p_2\equiv 1 \pmod{p_1}$. Similarly $p_1\equiv 1 \pmod{p_2}$, contrary to the former. Likewise (19_2) gives $p_3\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_3}$ and is excluded.

Next, let one triple be of the type (1, 1, 3) and two of the type (1, 2, 2); it suffices to treat

(20) $(p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5;\ p_3,\ p_4,\ p_1p_2p_5,$ or $p_4,\ p_5,\ p_1p_2p_3.$

But, in either case, we find that $p_4 \equiv 1 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_4}$.

Hence all three triples must be of type (1, 2, 2). Two of them may be taken to be the first two in (20), with $p_3 \neq p_4$, $p_2 \neq p_5$, $p_1 \neq p_5$. Since these are unaltered by (12)(34), the third may be restricted to one of the three:

$$p_3,\ p_1p_5,\ p_2p_4;\ p_5,\ p_1p_2,\ p_3p_4;\ p_5,\ p_1p_3,\ p_2p_4.$$

The first case gives $p_3 \equiv 1 \pmod{p_2}$, $p_2 \equiv 1 \pmod{p_3}$. The second case gives $p_2 \equiv 1 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_2}$. In the third case we may set $p_2 > p_1$; then

$$p_1p_4p_5 \equiv p_1p_2p_4 \equiv p_2p_4p_5 \pmod{p_3}, \quad p_1p_4p_5 \equiv p_1p_3p_4p_5 \equiv p_1p_3p_5 \pmod{p_2}.$$

If $p_3 = p_5$, then $p_1p_2p_4$ is a multiple of p_3, whereas $p_1 \neq p_5$, $p_2 \neq p_5$, $p_4 \neq p_3$. Hence $p_4p_5(p_2 - p_1) \equiv 0$, gives $p_2 - p_1 = cp_3$, $c > 0$. Similarly, $p_1p_4p_5(p_3 - 1) \equiv 0 \pmod{p_2}$ gives $p_3 \equiv 1 \pmod{p_2}$, since $p_4 = p_2$ would require $p_1p_3p_5 \equiv 0 \pmod{p_2}$. Hence $-p_1 \equiv c \pmod{p_2}$, so that $c \geq p_2 - p_1$. The former equation $p_2 - p_1 = cp_3$ is thus impossible in view of $p_3 > 1$. *There is no set of three triples involving fewer than six primes.*

8. Consider three triples involving six primes. If all are of type (1, 1, 4), they may be taken to be

(21) $$p_1,\ p_2,\ p_3p_4p_5p_6;\ p_3,\ p_4,\ p_1p_2p_5p_6;\ p_5,\ p_6,\ p_1p_2p_3p_4,$$

with $p_3p_4 > p_5p_6$. By $S_2 = S_3$, $p_3p_4 = p_5p_6 + \delta\, p_1p_2$, $\delta > 0$. But $p_1p_2 \equiv p_5p_6 \pmod{p_3p_4}$ by $S_1 = S_3$. Hence $0 \equiv p_5p_6(1 + \delta) \pmod{p_3p_4}$. Thus $\delta = -1 + \epsilon\, p_3p_4$, $\epsilon > 0$,

$$p_3p_4 + p_1p_2 = p_5p_6 + \epsilon\, p_1p_2p_3p_4.$$

But $p_1p_3 \cdot p_2p_4 \geq (p_1 + p_3)(p_2 + p_4) > p_1p_2 + p_3p_4$. Hence this case is excluded.

Let two of the triples be the first two in (21) and the third of type (1, 2, 3). Since the former are unaltered or interchanged by (12), (34), (56), (13)(24), the third may be assumed to be $p_5,\ p_1p_2,\ p_3p_4p_6$; or $p_5,\ p_1p_3,\ p_2p_4p_6$; or $p_5,\ p_1p_6,\ p_2p_3p_4$. The first yields $p_2 \equiv 1 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_2}$; the second, $p_3 \equiv 1 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_3}$. For the third case, $S_1 = S_2$ gives

(22) $p_1p_2=p_3p_4+\delta p_5p_6, \quad \delta+p_1p_3p_4+p_2p_3p_4=p_1p_2p_3+p_1p_3p_4,$

while $S_1=S_3$ gives $p_5p_6\equiv p_2 \pmod{p_3p_4}$. Thus $\delta\equiv p_1 \pmod{p_3p_4}$, by (22_1). Then (22_2) gives $p_1\equiv p_1p_2(p_3+p_4)$, mod p_3p_4. Hence

(23) $$1=p_2(p_3+p_4)+lp_3p_4 \quad (l \text{ negative integer}).$$

But $p_1p_3p_4p_5p_6\equiv p_3p_4\equiv p_1p_5p_6 \pmod{p_2}$. Thus $p_1p_5p_6\equiv 1$, $p_3p_4\equiv 1 \pmod{p_2}$. Hence in (23), $1\equiv l$, $l=1-mp_2$, $m>0$. Thus (23) may be given the form

$$p_2[(m-1)p_3p_4-p_3-p_4]+p_2p_3p_4=p_3p_4-1.$$

But if $m>1$, the quantity in brackets is positive and the equation impossible. Hence $m=1$, and (23) becomes

(24) $$p_2(p_3p_4-p_3-p_4)=p_3p_4-1.$$

Since $p_2\geq 2$, the first member exceeds the second if $(p_3-2)(p_4-2)>3$. Since (34) leaves each triple unaltered, we may set $p_3\geq p_4$. Hence, there remains the cases

$$p_3=5,\ p_4=3;\ p_3=p_4=3;\ p_4=2.$$

For $p_3=5$, $p_4=3$, we have $p_2=2$, $S_2=15+16p_1p_5p_6$, $S_3=30p_5+30p_1p_6+p_1p_5p_6$. Hence $S_2=S_3$ gives $(p_1p_6-2)(p_5-2)=3$, $p_1p_6=5$ or 3, which is impossible. For $p_3=p_4=3$, (24) gives $3p_2=8$. Hence must $p_4=2$, $p_2(p_3-2)=2p_3-1$. Thus $p_2\geq 3$, and the first member exceeds the second if $p_3>5$. Hence $p_3=5$, $p_2=3$, or $p_3=3$, $p_2=5$. For the first, $2S_2=2S_3$ becomes $(2p_5-3)(2p_1p_6-3)=7$, whence $p_1p_6=5$ or 2. For the second, $4S_2=4S_3$ becomes $(4p_1p_6-5)(4p_5-5)=21$, whence $p_1p_6=3$ or 2.

Next, let two of the triples be the first two in (21) and the third of type (2, 2, 2). The latter may be assumed to be p_1p_2, p_3p_4, p_5p_6; p_1p_2, p_3p_5, p_4p_6; p_1p_3, p_2p_4, p_5p_6; or p_1p_3, p_2p_5, p_4p_6. The first two and the last two are excluded by the argument excluding the first and second cases, respectively, at the beginning of the preceding paragraph.

Hence at most one of the three triples is of the type (1, 1, 4).

9. Let two triples be p_1, p_2, $p_3p_4p_5p_6$ and p_3, p_1p_2, $p_4p_5p_6$. Adding $p_1\ldots p_6$ to S_1 and S_2 and equating the sums, we get

(25) $$p_1p_2(p_3-1)(p_4p_5p_6-1)=p_3p_4p_5p_6(p_1-1)(p_2-1).$$

We may set $p_2 \geq p_1$, $p_2 \neq p_3$. Hence p_2 divides $p_4 p_5 p_6$, so that we may set $p_4 = p_2$. Since $p_4 p_5 p_6 - 1 > p_5 p_6 (p_2 - 1)$, we have $p_1 (p_3 - 1) < p_3 (p_1 - 1)$, $p_1 > p_3$. Hence $p_2 > p_3$. If the third triple is of type (1, 2, 3), it may be taken to be p_5, $p_1 p_3$, $p_2^2 p_6$; p_5, $p_1 p_6$, $p_2^2 p_3$; p_5, $p_2 p_3$, $p_1 p_2 p_6$; p_5, p_2^2, $p_1 p_3 p_6$; p_5, $p_2 p_6$, $p_1 p_2 p_3$; or p_5, $p_3 p_6$, $p_1 p_2^2$. Now S_1 is a multiple of p_2, so that S_3 must be. Hence for the first case, $p_1 p_3 p_5$ is a multiple of p_2, whence $p_1 = p_2$. Then S_1 is a multiple of p_2^2, so that S_3 must be; hence $p_3 p_5 \equiv 0 \pmod{p_2}$, which is impossible. For the second case, $p_1 p_5 p_6 \equiv 0 \pmod{p_2}$. If $p_1 = p_2$, $S_1 \equiv 0 \pmod{p_2^2}$, so that by S_3, $p_6 = p_2$. In any event, $p_6 = p_2$ and the second and third triples have $p_1 p_2 = p_1 p_6$ in common. For the third case, $S_1 = S_2$ gives $p_1 p_3 = p_1 + \epsilon\, p_5 p_6$, whence $\epsilon = \delta\, p_1$; while $S_1 = S_3$ gives $p_1 p_2 \equiv p_1 p_2 p_5 p_6$, $p_5 p_6 \equiv 1 \pmod{p_3}$, $p_5 p_6 > p_3$. For the fourth case, $S_1 \equiv S_3 \bmod p_2$ gives $p_1 p_3 p_6 p_5 \equiv 0 \pmod{p_2}$. If $p_1 = p_2$ the second and third triples would have a common element. Hence $p_2 = p_6$. By (25), $p_5 p_6$ must divide $p_1 (p_3 - 1)$. But $p_2 > p_3 - 1$. Hence $p_2 = p_6$ requires $p_1 = p_2$. For the fifth case, $p_1 \neq p_6$ by $p_2 p_6$ and $p_1 p_2$. Removing the factor p_2 from the S's, we see that $p_2 p_3 p_5 p_6 \equiv p_3 p_5 p_6 \equiv p_5 p_6 \pmod{p_1}$, $p_3 \equiv 1 \pmod{p_1}$, contrary to $p_3 < p_1$. For the sixth case, $S_1 \equiv S_3 \pmod{p_2}$ gives $p_3 p_5 p_6 \equiv 0 \pmod{p_2}$, $p_6 = p_2$. As in the fourth case, $p_1 = p_2$. Then S_1, but not S_3, is a multiple of p_2^2.

If the third triple is of type (2, 2, 2), it may be taken to be

$$p_2^2,\ p_1 p_3,\ p_5 p_6;\quad p_2^2,\ p_1 p_5,\ p_3 p_6;\quad p_2 p_3,\ p_2 p_5,\ p_1 p_6;\quad \text{or } p_2 p_5,\ p_2 p_6,\ p_1 p_3.$$

For the first two cases, $p_1 \neq p_2$ and $S_1 \equiv S_3 \pmod{p_2}$ gives $p_6 = p_2$, contrary to the above. The third case is excluded by $p_3 \equiv 1 \pmod{p_5 p_6}$, $p_5 p_6 \equiv 1 \pmod{p_3}$; the fourth by $p_1 \equiv 1 \pmod{p_3}$, $p_3 \equiv 1 \pmod{p_1}$.

THE UNIVERSITY OF CHICAGO, *May 16, 1909.*

NOTE ON SOME POLYNOMIALS RELATED TO LEGENDRE'S COEFFICIENTS.*

By R. D. CARMICHAEL, Anniston, Alabama.

The object of this note is to point out some properties of a class of functions which contains Legendre's coefficients as a special case. It will be seen that the former possess some interesting properties belonging to the latter.

1. Consider the definite integral†

*Presented to the American Mathematical Society, April 24, 1909.

†Cf. Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 173.

$$\int_a^b Q\frac{d^n R}{d x^n}dx=0,$$

where Q is any polynomial of degree less than n. It is required to find a polynomial d^nR/dx^n of degree $n(2\alpha-1)$, α odd, satisfying the above equation. Since we desire only the value of d^nR/dx^n we may choose n zeros of R at will. Let each of them be a, so that $(x-a)^n$ is a factor of R. Now, integrating by parts, we have

$$\int_a^b Q\frac{d^n R}{dx^n}dx\left[QR^{(n-1)}-Q'R^{(n-2)}+Q''R^{(n-3)}-\ldots\pm Q^{(n-1)}R\right]_a^b,$$

where the indices in the brackets indicate differentiation with respect to x. Since $(x-a)^n$ is a factor of R,

$$R(a)=0,\ R'(a)=0,\ R''(a)=0,\ \ldots,\ R^{(n-1)}(a)=0;$$

and therefore,

$$Q(b)R^{(n-1)}(b)-Q'(b)R^{(n-2)}(b)+\ldots\pm Q^{(n-1)}(b)R(b)=0;$$

or, since Q is arbitrary by hypothesis,

$$R^{(n-1)}(b)=0,\ R^{(n-2)}(b)=0,\ \ldots,\ R(b)=0.$$

Hence $(x-b)^n$ is a factor of R; that is,

$$R=c(x-a)^n(x-b)^nf(x),$$

where $f(x)$ is an arbitrary polynomial of degree $2n(\alpha-1)$ and c is a constant. Hence,

$$\int_a^b Q\frac{d^n}{dx^n}[c(x-a)^n(x-b)^nf(x)]dx=0, \tag{1}$$

where Q is any polynomial of degree less than n and $f(x)$ is any polynomial of degree $2n(\alpha-1)$.

Since $f(x)$ is of degree $2n(\alpha-1)$, α odd,* it is evident that we may choose $f(x)$ so that

$$R=c(x^\alpha-a^\alpha)^n(x^\alpha-b^\alpha)^n,$$

and therefore

$$\int_a^b Q\frac{d^n}{dx^n}[c(x^\alpha-a^\alpha)^n(x^\alpha-b^\alpha)^n]dx=0. \tag{2}$$

*This is the first use of the fact that α is odd.

Now take $a=-1$, $b=1$, $c=\dfrac{1}{2^n.\,n!}$, and set

$$P_{na}=\frac{1}{2^n.\,n!}\frac{d^n}{dx^n}(x^{2a}-1)^n,$$

so that $P_{n1}=P_n$ in the ordinary notation for Legendre's coefficients. Then by (2),

$$\int_{-1}^{+1}QP_{na}dx=0,$$

where Q is any polynomial of degree less than n. In particular,

$$\int_{-1}^{+1}P_{m1}P_{na}dx=0,\ m<n,$$

a generalization of the formula $\int_{-1}^{+1}P_mP_ndx=0,\ m\neq n$.

If we write P_{na} in the form

$$P_{na}=\frac{1}{2^n.\,n!}\frac{d^n}{dx^n}(x^a-1)^n(x^a+1)^n,$$

from the indicated nth derivative of the product by Leibniz' formula, and in the result give to x the values $+2$ and -1, we have

$$P_{na}(1)=a^n,\ P_{na}(-1)=(-a)^n,$$

a generalization of the corresponding formulae for Legendre's coefficients.

Again, since two real roots of $\dfrac{d^s}{dx^s}(x^a-1)^n(x^a+1)^n$, $s<n$, are -1 and $+1$, it follows from Rolle's Theorem that P_{na} has n real roots between -1 and $+1$.

2. If x, y, z are connected by the relation

$$z=x+y\phi(z),$$

then by Lagrange's formula we have

$$f(z)=f(x)+y\phi(x)f'(x)+\frac{y^2}{2!}\frac{d}{dx}[\phi(x^2)f'(x)]+\ldots$$

$$+\frac{y^{n-1}}{(n-1)!}\frac{d^{n-2}}{dx^{n-2}}[\phi(x)^{n-1}f'(x)]+\ldots$$

Therefore if we assume the special relation

$$z=x+\frac{y}{2}(z^{2a}-1),$$

we may write the value of z in the form

$$z=x+\frac{y}{2}(x^{2a}-1)+\frac{1}{2!}\left(\frac{y}{2}\right)^2\frac{d}{dx}(x^{2a}-1)^2+\frac{1}{3!}\left(\frac{y}{2}\right)^3\frac{d^2}{dx^2}(x^{2a}-1)^3+\ldots$$
$$+\frac{1}{n!}\left(\frac{y}{2}\right)^n\frac{d^{n-1}}{dx^{n-1}}(x^{2a}-1)^n+\ldots$$

Differentiating with respect to x, regarding y as constant, we have the formula

$$\frac{dz}{dx}=1+\frac{y}{2}\frac{d}{dx}(x^{2a}-1)+\frac{1}{2!}\left(\frac{y}{2}\right)^2\frac{d^2}{dx^2}(x^{2a}-1)^2+\ldots$$
$$+\frac{1}{n!}\left(\frac{y}{2}\right)^n\frac{d^n}{dx^n}(x^{2a}-1)^n+\ldots;$$

or, defining $P_{0a}=1$, this expansion may be written,

$$\frac{dz}{dx}=P_{0a}+yP_{1a}+y^2P_{2a}=y^3P_{3a}+\ldots$$

an equation which holds subject to the sole condition that x, y, z are connected by the relation (3). If $a=1$, dz/dx becomes $(1-2xy+y^2)^{-\frac{1}{2}}$ as may be easily shown by solving (2) for z and differentiating the result with respect to x; and therefore we have the well-known expansion of $(1-2xy+y^2)^{-\frac{1}{2}}$ in ascending powers of y, the coefficients being Legendre's polynomials.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

313. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the equations $px^2+qx+r=0$ and $\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0$ may give equal values for y.

Solution by G. W. HARTWELL, University of Kansas, Lawrence, Kansas.

$$px^2+qx+r=0\ldots(1).$$

$$\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0\ldots(2).$$

Since (2) is of the first degree in y, the values of y will be equal when the roots of (1) are equal.

$\therefore q^2-4pr=0$ is one condition that (1) and (2) give equal values of y.

If $q^2-4pr\neq 0$, we have on solving (1) and substituting in (2),

$$y=-\frac{\alpha q^2-2\alpha pr-\beta pq+2\gamma p^2\pm(\beta p-\alpha q)\sqrt{(q^2-4pr)}}{aq^2-2apr-bpq+2cp^2\pm(bp-aq)\sqrt{(q^2-4pr)}}\ldots(3).$$

The two values given by (3) will be equal when,

$$(bp-aq)(\alpha q^2-2\alpha pr-\beta pq+2\gamma p^2)$$
$$-(\beta p-\alpha q)(aq^2-2apr-bpq+2cp^2)=0\ldots(4).$$

(4) can be put in the form:

$$p^2\begin{vmatrix} p & q & r \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}=0.$$

$\therefore$ The conditions that (1) and (2) give equal values of y are

I. $q^2-4pr=0$. Or, II. $\begin{vmatrix} p, & q, & r \\ a, & b, & c \\ \alpha, & \beta, & \gamma \end{vmatrix}=0$. Or, III. $p=0$.

In this case (1) becomes an equation of the first degree, and hence (2) can give only one value of y.

Also solved by V. M. Spunar.

314. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Sum to infinity the series $\frac{1}{2.3.3.4}+\frac{1}{4.5.5.6}+\frac{1}{6.7.7.8}+\frac{1}{8.9.9.10}+\cdots$

I. Solution by E. B. ESCOTT, Ann Arbor, Mich., and G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The general term of the series is $\frac{1}{2n(2n+1)^2(2n+2)}$, which may be resolved into the partial fractions,

$$\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{(2n+1)^2}\Big).$$

Therefore,

$$\frac{1}{2.3.3.5}=\frac{1}{4}\left(1-\frac{1}{2}\right)-\frac{1}{3^2}$$

$$\frac{1}{4.5.5.6}=\frac{1}{4}\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{5^2}$$

$$\frac{1}{6.7.7.8}=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4}\right)-\frac{1}{7^2}$$

. . . . . .

Adding, we have

$$\frac{1}{2.3.3.4}+\frac{1}{4.5.5.6}+\cdots=\frac{1}{4}-\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots.\right)=\frac{1}{4}-\left(\frac{\pi^2}{8}-1\right).$$

[See E. Pascal, *Repertorium der hoeheren Mathematik*, Vol. 1, p. 60.]

The sum is, therefore, $\frac{5}{4}-\pi^2/8=.0163$.

II. Solution by S. A. COREY, Hiteman, Iowa.

Take the Fourier's cosine series for $x\sin x$, viz.,

$$x\sin x=1-\frac{\cos x}{2}-\frac{2\cos 2x}{1.3}+\frac{2\cos 3x}{2.4}-\frac{2\cos 4x}{3.5}+\cdots.$$

and integrate both members of the equation three times to obtain the only constant of integration involved, viz., $(\frac{1}{2}+\pi^2/12)$, in the series obtained by the second integration. The latter series may then be thus written,

$$\frac{-x\sin x}{2}-\frac{5\cos x}{4}-\frac{x^2}{4}+\frac{1}{2}+\frac{\pi^2}{12}=\frac{\cos 2x}{1.2.2.3}-\frac{\cos 3x}{2.3.3.4}+\frac{\cos 4x}{3.4.4.5}-\cdots(1).$$

When $x=0$, (1) becomes

$$\frac{1}{1.2.2.3}-\frac{1}{2.3.3.4}+\frac{1}{3.4.4.5}-\frac{1}{4.5.5.6}+\ldots=\frac{\pi^2}{12}-\frac{3}{4}\ldots(3).$$

When $x=\pi$, (1) becomes

$$\frac{1}{1.2.2.3}+\frac{1}{2.3.3.4}+\frac{1}{3.4.4.5}+\frac{1}{4.5.5.6}+\ldots=\frac{7}{4}-\frac{\pi^2}{6}\ldots(4).$$

Subtracting (3) from (4), we find the sum of the given series to be $\frac{5}{4}-\pi^2/8$.

Also solved by J. Scheffer.

315. Proposed by PROFESSOR B. F. YANNEY, Mount Union College, Alliance, Ohio.

Simplify, $1-(2-(3-\ldots-(n-1)-n)\ldots)))$.

Solution by GEORGE W. HARTWELL, University of Kansas, Lawrence, Kansas, and V. M. SPUNAR, Pittsburg, Pa.

Removing the parentheses, this expression becomes

$$1-2+3-4+\ldots(-1)^{n-1}n\equiv\sum_1^n(-1)^{n-1}n.$$

But $\sum_1^n(-1)^{n-1}n=-(n/2)$ when n is even,

and $\sum_1^n(-1)^{n-1}n=(n+1)/2$ when n is odd.

Also solved by G. B. M. Zerr.

GEOMETRY.

342. Proposed by G. I. HOPKINS, M. A., Instructor in Mathematics and Astronomy, Manchester, N. H.

Given, circle DEF inscribed in triangle ABC and circumscribing the triangle DEF, D, E, F being the points of contact; AH is drawn through center, N, meeting chord DF in H. Through H is drawn BK meeting AC in K. Prove triangle ABK isosceles.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let the points D, F, E be situated on the sides a, b, c, respectively, and also let $l=\cos^2(A/2)$, $m=\cos^2(B/2)$, $n=\cos^2(C/2)$. Then $(0, rn, rm)$; $(rn, 0, rl)$, are the trilinear coordinates of D and F, respectively.

Hence $\beta-\gamma=0$ is the equation to AN, $l\alpha+m\beta-n\gamma=0$ is the equation to DF.

$\therefore (n-m,\ l,\ l)$; $(0,\ 2\triangle/b,\ 0)$, are the coordinates of H and B, respectively. $\therefore l\alpha+(m-n)\gamma=0$ is the equation to BK.

But $m-n+l\cos C+(m-n)\cos A-l\cos B=0$.

$\therefore BK$ is perpendicular to AN. $\therefore$ triangle ABK is isosceles.

II. Solution by G. I. HOPKINS, Instructor in Mathematics and Astronomy, High School, Manchester, N. H.

Construction: Join HE, HB. Extend DE and draw BP perpendicular to it.

Demonstration: Since BP and AH are perpendicular to DE, they are parallel. $AD=AE$, *i. e.*, $\triangle ADE$ is isosceles. arc $ES=$arc SF.

$\therefore \angle ENB=\angle EDF$. $\therefore \triangle DRH$ and $\triangle NEB$ are similar.

$\therefore NE/EB=DR/RH$. $\angle NEB$ is a right angle.

$\therefore \angle REN$ is complement to $\angle BEP$. $\therefore \angle REN=\angle EBP$.

$\therefore \triangle REN$ and $\triangle EBP$ are similar.

$\therefore RE/BP=NE/EB$; $\therefore DR/RH=RE/BP$.

Since $DR=RE$, $\therefore RH=BP$, and $\therefore RHBP$ is a parallelogram; *i, e.*, BK is parallel to DE.

$\therefore \triangle ABK$ is similar to $\triangle ADE$, and is therefore isosceles.

CALCULUS.

270. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=0}^{x=\infty} \frac{1}{(a^2+x^2)^n}=\frac{\pi}{2a^{2n-1}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}+\frac{1}{2a^{2n}}$, n being a positive integer >1.

II. Solution by the PROPOSER.

Performing the finite summation of the problem by the aid of Maclaurin's Summation-formula,

$$\Sigma u_x=C+\int u_x dx-\tfrac{1}{2}u_x+\frac{B_1}{2!}\frac{du_x}{dx}-\frac{B_2}{4!}\frac{d^3u_x}{dx^3}+\ldots$$

(See Boole's *Finite Differences*, page 90), we readily obtain the above expression for the sum, if we substitute for the definite integral, $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$, its well known value, $\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\cdot\frac{1}{a^{2n-1}}$, if n is a positive integer >1.

The solution in the May MONTHLY involves the erroneous assumption that $\sum_{x=0}^{x=\infty} \frac{1}{(a^2+x^2)^n}=\int_0^\infty \frac{dx}{(a^2+x^2)^n}$.

271. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-5\left(\frac{d^2y}{dx^2}\right)\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\tfrac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

show that there is an integrating factor of the form $\left(\frac{d^2y}{dx^2}\right)^n$, and integrate the equation.

Solution by the PROPOSER, and LEVI S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

When the equation is multiplied by $(d^2y/dx^2)^n$, if it is an exact equation, its first integral is

$$\left(\frac{d^2y}{dx^2}\right)^{n+2}\frac{d^4y}{dx^4}+\frac{40}{9(n+1)}\left(\frac{d^2y}{dx^2}\right)^{n+1}\left(\frac{d^3y}{dx^3}\right)^2=c_1\ldots(2).$$

Differentiating this equation and comparing with original equation, we find that $n=-\frac{11}{3}$ and $-\frac{13}{3}$.

In (2), put $n=-\frac{11}{3}$. The equation is exact and its integral is

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{5}{3}}\frac{d^3y}{dx^3}=c_1x+c_2.$$

Integrating again,

$$-\tfrac{3}{2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=\frac{c_1}{2}x^2+c_2x+c_3.$$

Solving for d^2y/dx^2, and changing the constants,

$$\frac{d^2y}{dx^2}=\frac{1}{(c_1x^2+2c_2x+c_3)^{\frac{3}{2}}}.$$

Integrating twice, we have

$$y=\frac{1}{c_1c_3-c_2^2}\sqrt{(c_1x^2+2c_2x+c_3)}+c_4x+c_5.$$

Also solved by G. B. M. Zerr, and V. M. Spunar.

272. Proposed by CLARENCE OHLENDORF, Chicago, Ill.

Find $\int \log_e \tan^{-1} x\,dx$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and C. N. SCHMALL, New York City.

Putting $x=\tan y$, we have

$$\int \log \tan^{-1}x\,dx=\int \log y\,\frac{dy}{\cos^2 y}=\int \log y\sec^2 y\,dy=\log y-\tan y-\int\frac{\tan y\,dy}{y}.$$

Substituting for $\tan y$ its series, and integrating, we get

$$\int \frac{\tan y\,dy}{y} = \frac{2^2(2^2-1)}{2!}B_1 y + \frac{2^4(2^4-1)}{4!}B_3\frac{y^3}{3} + \frac{2^6(2^6-1)}{6!}B_5\frac{y^5}{5},$$

where B_1, B_3, B_5, ... are Bernoulli's numbers.

Therefore the given integral is

$$x\log\tan^{-1}x - \left[\frac{2^2(2^2-1)}{2!}B_1\tan^{-1}x + \frac{2^4(2^4-1)}{4!}B_3\frac{(\tan^{-1}x)^3}{3} + \ldots\right].$$

Also solved by G. B. M. Zerr, V. M. Spunar, and J. E. Sanders.

273. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both begin to swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If the line joining the positions of the dog and the duck is tangent to the dog's path, the following is the solution.

Let A, B be the starting points of dog and duck; O, the center of the pond; P, R, corresponding positions of the dog and duck. Then the tangent t, from P must pass through R. Let the duck swim to S and the dog to Q. Draw OF, PH perpendicular to AB, PI, QD perpendicular to OF, QC perpendicular to PI, and RE perpendicular to QS, produced.

Let $AB=2a$, $AP=\sigma$, $\angle RAB=\phi$, $OD=x$, $n/m=b$. Then $PR=d\sigma$, $\angle RAS=d\phi$, $DI=QC=dx$.

Now $\sigma=2ab\phi$; $\therefore d\sigma/2ad\phi=b$.

In the limit the triangles PQC and RES are similar.

$\therefore d\sigma : dx=2ud\varphi : ES$. $\therefore ES=(2ad\phi/d\sigma)dx=dx/b$.

The tangent t has negative increments at both ends (PQ and ES).

$\therefore dt=-d\sigma-dx/b$, or $t=C-\sigma-x/b$. When $t=2a$, $\sigma=0$, $x=0$.

$\therefore C=2a$. $\therefore t=2a-2ab\phi-x/b$. When $t=0$, $x=2a\sin\phi\cos\phi$.

$\therefore b=b^2\phi+\sin\phi\cos\phi=(b^2+1)\phi-\frac{2}{3}\phi^3+\frac{2}{15}\phi^5-\frac{4}{315}\phi^7+\ldots$

$$Q=\frac{b}{b^2+1}+\frac{2b^4}{3(b^2+1)^4}+\frac{2b^5(9-b^2)}{15(b^2+1)^7}+\frac{4(225+b^4-54b^2)b^7}{315(b^2+1)^{10}}+\ldots$$

by reversion of series.

b. If the dog always keeps on the line joining his starting point to the duck's position, the solution becomes quite simple.

Then $us=2a\psi$ is the intrinsic equation to the dog's path.

$$\therefore \frac{ds}{d\psi}=\frac{2a}{u}=\sqrt{r^2+\left(\frac{dr}{d\psi}\right)^2}. \therefore d\psi=\frac{dr}{\sqrt{\frac{4a^2}{u^2}-r^2}}, \text{ and } \psi=\sin^{-1}\frac{ur}{2a}.$$

When $r=2a\cos\psi$ the dog catches the duck.

$\therefore \psi=\sin^{-1}(u\cos\psi)$, or $\tan\psi=u$, and $\psi=\tan^{-1}u$.

$\therefore us=2a\tan^{-1}u$, or $s=(2a/u)\tan^{-1}u=(2an/m)\tan^{-1}(m/n)$ is the distance the dog swims to overtake the duck.

MECHANICS.

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225. Proposed by W. A. BALDWIN, Springfield, Mo.

Find, *by means of polar coordinates*, the moment of inertia about the origin of the area between the parabola $ay=2(a^2-x^2)$, the circle $x^2+y^2=a^2$, and the axis of Y.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The polar equation to the parabola is

$$r=\frac{a[\sqrt{(1+15\cos^2\theta)}-\sin\theta]}{4\cos^2\theta}=r_1.$$

The polar equation to the circle is $r=a$.

$\therefore m$, moment of inertia required,

$$=\int_{\pi/6}^{\pi/2}\int_a^{r_1} r^3 d\theta\, dr=\frac{a^4}{1024}\int_{\pi/6}^{\pi/2}[8\sec^8\theta+112\sec^6\theta+136\sec^4\theta-256$$

$$-8\sin\theta\sec^6\theta(7+\sec^2\theta)\sqrt{(1+15\cos^2\theta}]d\theta=\frac{a^4}{4}\left(\frac{883\sqrt{3}}{280}-\frac{\pi}{3}\right).$$

This is problem 7, p. 350, Osborne's *Differential and Integral Calculus*. Osborne gives as the result, $(\frac{53}{56}-\frac{1}{6}\pi)a^4$. Professor Zerr's result is correct. ED. F.

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THE EVOLUTION OF MODERN NUMERALS FROM ANCIENT TALLY MARKS.

By MAJOR CHARLES E. WOODRUFF, A. M., M. D., U. S. Army.

No acceptable explanation has ever been given as to the origin of our numerals though the literature on the subject is enormous. The multiplicity of the theories is evidence of the inadequacy of each. Of course it is known that the Arabs obtained the signs from Southern Asia or India before the ninth century A. D. and brought them to Europe in the tenth, but the remote origin has never been discovered. The later Greeks and Hebrews used the first letters of their respective alphabets to represent units, the second groups to represent tens and the third for the hundreds. Sometimes the initial letter of the word for the number was used as a symbol for that number, as in the early Greek, and possibly the Roman C and M, though the latter may have been evolved from earlier symbols. For these reasons quite a number of scholars have sought for the original forms of our numerals in the letters of some alphabet. There is certainly quite a remarkable resemblance between some of the old numerals and some of the letters of old alphabets, but that is no proof of common origin. Indeed it would be strange if we could not find many such coincidences in the innumerable forms which both letters and numerals have taken. Indeed numerals were used where there were no alphabets or before alphabets were evolved as in modern China and ancient Egypt.

In addition, Sir E. Clive Bagley and George Bühler point out the fact (*Journal Royal Asiatic Society*, p. 335, 1882) that there is no known reason why certain letters should have been selected to represent the numerals four to nine, which the former accepts as being derived from letters from several alphabets widely separated. Bagley curiously enough also states that all numerals at first were merely shorthand ways of expressing their names, which wholly contradicts the idea of derivation from letters, and ignores the fact that primitive tallies or numerals must have existed before they had names.

It has also been frequently asserted that the numerals were inventions